MATHEMATICS

# THE STRUCTURE OF A LINEAR TRANSFORMATION

BY

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## 1. *Introduction*

Throughout sections 1–5 of this article $T$ will denote a linear transformation on a vector space $V$ which is $n$-dimensional ($n$ finite) over a division ring $F$. We shall assume that $F$ is commutative [1]).

Texts on Algebra give a discussion of the structure of such a linear transformation $T$, leading to canonical matrix representations (see e.g. [3, Chapter III], [5, §§ 108–111]); the discussion is usually in terms of invariant factors, elementary divisors and cyclic subspaces (or cyclic submodules).

We give here a somewhat different proof of the structure theorem, designed to permit extensions given in section 6 as well as an extension to the case of general von Neumann geometries (see [2]).

With a well known argument we first reduce the discussion to the special case that $q(T)=0$ where $q$ is a power of some irreducible polynomial $p$ [2]). Then we discuss separately the two cases:

$$\text{(i)} \quad p(t)=t, \qquad \text{(ii)} \quad p(t) \text{ different from } t,$$

using lattice theoretic arguments applied to the lattice of subspaces of $V$ [3]).

For case (i), that is, $T$ nilpotent, there is no particular difficulty, but for case (ii) there is one complication which we point out explicitly in the Remark following Lemma 1 of section 5.

## 2. *Preliminaries*

We shall use the following notation:

(i) $M_1 + \ldots + M_r$ will denote the vector sum of the subspaces $M_1, \ldots, M_r$; this coincides with the lattice union (see footnote 3)). We write $\oplus$

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1) In other words, $F$ is a (commutative) field.

2) In this article polynomial will mean: polynomial with coefficients in $F$; $p$ irreducible will mean: $p$ is of degree $\geqslant 1$, $p$ has leading coefficient 1 and $p$ cannot be expressed as a product of two polynomials each of degree $\geqslant 1$.

3) As is well known, this lattice is modular, complemented (hence relatively complemented), complete and satisfies von Neumann's lattice continuity conditions (see [2]). This lattice is also atomic (the subspace spanned by a single non-zero vector is an atom) but arguments which use this fact may not extend to the non-atomic case.

to imply that the subspaces are *independent*, that is, $M_j \cap (M_1 + \dots$ $\dots + M_{j-1} + M_{j+1} + \dots + M_r) = 0$ for $j = 1, \dots, r$, equivalently:

$$M_j \cap (M_1 + \dots + M_{j-1}) = 0 \text{ for } j = 2, \dots, r;$$

$[v]$ will denote the subspace spanned by the vector $v$.

(ii) If $M \supset N$ then $[M-N]$ will denote any (fixed) subspace such that $[M-N] \oplus N = M$.

(iii) $D(M)$ will denote the dimension of the subspace $M$ (non-empty by definition), thus $D(0)=0, \dots, D(V)=n$.

(iv) $R(T)$ will denote the rank of $T$ $(=D(TV))$, thus $R(0)=0$, $R(1)=n$.

(v) $f(p, s, T)$ will denote the *multiplicity function*:

$$\frac{1}{\text{degree of } p}\,((R(p^{s-1}(T)) - R(p^s(T))) - (R(p^s(T)) - R(p^{s+1}(T))))$$

defined for each polynomial $p$ of degree $\geqslant 1$ and each integer $s \geqslant 1$.

(vi) $N(T)$ will denote the null space of $T$, that is $\{v \mid Tv=0\}$.

(vii) $N_p = N_p(T)$ will denote the subspace of all $v$ for which $p^s(T)v=0$ for some s.

We recall some well known simple facts:

(viii) $R(T) + D(N(T)) = n$.

(ix) $R(TS) \leqslant R(T)$, $D(N(TS)) \geqslant D(N(S))$ obviously; hence, using (viii), $R(TS) \leqslant R(S)$, $D(N(TS)) \geqslant D(N(T))$.

(x) $N_p = N(p^n(T))$. For suppose $v$ is a vector in $N_p$. Let $S = p(T)$ and suppose $s$ is the least exponent for which $S^s v = 0$. Then $S^i v = \sum(c_j S^j v \mid 0 \leqslant j < i)$ for some $i \leqslant n$. Successive left multiplication by $S^{s-k}$, $k = 1, 2, \dots$ shows, for $j = 0, \dots, i-1$ in turn, that $c_j S^{s-1} v = 0$, $c_j = 0$. Hence $S^i v = 0$. Since $i \leqslant n$ it follows that (x) holds.

We note also:

(xi) $$0 \leqslant f(p, s, T) \leqslant \frac{n}{\text{degree of } p}.$$

Let $S = p(T)$. Since $R(S^s) \leqslant R(S^{s-1}) \leqslant n$ we need only observe that

$$\begin{aligned} R(S^{s-1}) - R(S^s) &= D(\{v \mid v \in S^{s-1}V \text{ and } Sv = 0\}) \\ &\geqslant D(\{v \mid v \in S^s V \text{ and } Sv = 0\}) \\ &= R(S^s) - R(S^{s+1}). \end{aligned}$$

We shall use the following (known) facts:

(xii) $TM \subset M$ implies $p(T)M \subset M$ for all polynomials $p$ (obviously).

(xiii) $TM \subset M$, $q(T)M=0$, and $p$, $q$ relatively prime [1]) imply that the restriction of $p(T)$ to $M$ is invertible in $M$.
For: $u(t)p(t)+w(t)q(t)=1$ for suitable $u$, $w$. Hence on $M$, $u(T)p(T)=1$.

(xiv) $h=pq$ with $p$, $q$ relatively prime implies $N(h(T))=N(p(T)) \oplus N(q(T))$.
Obviously $\geqslant$ holds; by (xiii), $N(p(T)) \cap N(q(T))=0$; and for each vector $v$,
$v=u(T)p(T)v+w(T)q(T)v=v_1+v_2$ say, and if $h(T)v=0$ then $q(T)v_1=$ $=p(T)v_2=0$.

(xv) $V=\Sigma \oplus (N_p \mid p$ varying over all irreducible polynomials).
By (x), $N_p=N(p^n(T))$ so (xiv) implies $N_h=\Sigma \oplus (N_{p_i} \mid i=1, \ldots, r)$if $h$ is a product of different irreducible polynomials $p_1, \ldots, p_r$. Since $q_0(T)V=0$ for some polynomial $q_0$ of degree $\geqslant 1$ ($T^i$, $0 \leqslant i \leqslant n^2$ are linearly dependent), hence $V=N_h$ if $h$ is the product of those irreducible polynomials which are factors of $q_0$ and so (xv) holds.

Finally, we shall make essential use of the following concept of a *q-space* (more precisely, a *q-T*-space):

(xvi) A subspace $M \neq 0$ will be called a $q$-space if:
$q$ (of degree $m$, say) is a power of some irreducible polynomial, and $q(T)M=0$, and for some subspace $N$ (to be called a *q-base space for M*): $N$, $TN$, $\ldots$, $T^{m-1}N$ are equidimensional [2]) and $M=N \oplus TN \oplus \ldots$ $\ldots \oplus T^{m-1}N$.

## 3. *The structure theorem*

Theorem 1. *Suppose $T$ is a linear transformation in a vector space $V$ which is n-dimensional ($n$ finite) over a field $F$. Then:*

(a) *$V$ possesses a decomposition $V=M_1 \oplus \ldots \oplus M_r$ such that each $M_i$ is a $q_i$-space with all $q_i$ different;*

(b) *Whenever* (i) *holds then for each power of an irreducible polynomial $q=p^s$:*
*$q=q_i$ for some $i$ if and only if $f(p, s, T)>0$ and then* $D(M_i)=$(degree of $q_i$) $\times f(p, s, T)$.

(c) *A linear transformation $T_1$ can be expressed as $T_1=PTP^{-1}$ for some invertible linear transformation $P$ if and only if $f(p, s, T)=f(p, s, T_1)$ for all irreducible polynomials $p$ and integers $s \geqslant 1$* [3]).

Remark 1. Our Theorem 1 is, of course, very close to the usual theorem in Algebra texts.

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[1]) Polynomials $p$, $q$ are called relatively prime if they have no common factor of degree $\geqslant 1$ (in particular if $p$, $q$ are powers of different irreducibles).

[2]) This is equivalent to: $D(T^{m-1}N) = D(N)$, and also to: $T^{m-1}v \neq 0$ whenever $0 \neq v \in N$.

[3]) A simpler equivalent condition is: $R(p^s(T)) = R(p^s(T_1))$ for all irreducible polynomials $p$ and integers $s \geqslant 1$.

Suppose Theorem 1 proved and let $m_i$ denote the degree of $q_i$. Then each $M_i$ has a $q_i$-base space $N_i$. Choose any basis $v_{i,1}, \ldots, v_{i,f_i}$ for $N_i$

$$\left( f_i = \frac{D(M_i)}{m_i} \right)$$

Then for each $j (1 \leqslant j \leqslant f_i)$:

$$M_{i,j} \equiv [v_{i,j}] \oplus T[v_{i,j}] \oplus \ldots \oplus T^{m_i-1}[v_{i,j}]$$

is a $q_i$-space with $[v_{i,j}]$ as $q_i$-base space. The vectors $v_{i,j}, Tv_{i,j}, \ldots, T^{m_i-1}v_{i,j}$ are a basis for $M_{i,j}$ and relative to this basis, the restriction of $T$ to $M_{i,j}$ is (obviously) represented by the matrix $B(q_i)$, the companion matrix [1]) of $q_i$.

Taken together for all $1 \leqslant j \leqslant f_i$, $1 \leqslant i \leqslant r$, these bases for $M_{i,j}$ are a basis for $V$; relative to this basis $T$ is represented by the matrix which for each $i = 1, \ldots, r$, has $B(q_i)$ with multiplicity $f_i$, as blocks on the main diagonal.

Such a matrix of diagonal blocks $B(q_i)$ with each $q_i$ a power of an irreducible polynomial, is called in [5, page 121], the "second normal form" (see also [3, page 94] for a closely related matrix called there the "classical" canonical matrix). Texts usually refer to the $q_i$ (with multiplicity $f_i$) [2]) as the elementary divisors of $T$.

Proof of (c). Part (c) of Theorem 1 can be deduced from (a) and (b) of Theorem 1, as follows.

Since rank is preserved under an inner isomorphism: $T \to PTP^{-1}$ (use (ix)), the "only if" part of (c) is immediate.

On the other hand, if (a) and (b) hold then the equality $f(p, s, T) = f(p, s, T_1)$ for all irreducible $p$ and $s \geqslant 1$ implies that $T$ and $T_1$ have the same "second normal form of canonical matrix" (use the preceding Remark 1) and hence the "if part" of (c) also holds.

Thus to prove Theorem 1, we need only prove (a) and (b).

Proof of (b). Because of (xii) it is sufficient to verify (b) in each $M_j$ separately, so we may suppose $r = 1$, $V = M_1$, $q_1 = p_1^{s_1}$, $f_1 = \dfrac{n}{\text{degree of } q_1}$.

If $p \neq p_1$ then by (xiii), $p^j(T)$ is invertible for all $j$ so $f(p, s, T) = 0$, as required.

If $p = p_1$ we shall prove that

$$D(N(p^j(T))) = f_1 \times (\text{degree of } p) \times \min(j, s_1). \tag{1}$$

Then using (viii)), $f(p, s, T) = 0$ if $s + 1 \leqslant s_1$ or if $s_1 \geqslant s - 1$ and $= D(V)$ if $s = s_1$, which will establish (b).

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[1]) If $q(t)$ is any polynomial $t^m + c_{m-1} t^{m-1} + \ldots + c_0$ the companion matrix of $q$ is defined to be the $m \times m$ matrix $B(q) = (b_{i,j})$ which has $b_{i+1,i} = 1$ for $1 \leqslant i \leqslant m - 1$, $b_{i,m} = -c_{i-1}$ for $1 \leqslant i \leqslant m$ and all other $b_{i,j} = 0$.

[2]) Theorem 1 (b) will show that if $q_i = p_i^{s_i}$ then $f_i$ is uniquely determined as $\dfrac{D(M_i)}{\text{degree of } q_i} = f(p_i, s_i, T)$ (this motivated our terminology: multiplicity function).

To prove (1) it is clearly sufficient to show: if $p=p_1$ and $j<s_1$ then $D(N(p^j(T)))=f_1\times(\text{degree of } p)\times j$. But if $N_1$ is a $q_1$-base space for $M_1=V$, then $p^j(T)v\neq 0$ for $0\neq v\in N_1\oplus\ldots\oplus T^iN_1$ where $i+(\text{degree of } p)\times\times j=q-1$; hence

$$\begin{aligned} R(p^j(T)&\geqslant f_1\times(i+1)\\ &=f_1\times(\text{degree of } p)\times(s_1-j);\\ D(N(p^j(T)))&\leqslant f_1\times(\text{degree of } p)\times j. \end{aligned}$$

Since $p^{s_1}(T)=0$, necessarily $D(N(p^j(T)))\geqslant R(p^{s_1-j}(T))\geqslant f_1\times(\text{degree of } p)\times j$, so (1) does hold.

Remark 2. To prove (a) of Theorem 1 it is sufficient to prove this for each $N_p$ (in place of $V$), because of (xv). Thus the full strength of Theorem 1 will be established as soon as we prove Theorem 1(a) under the additional assumption: $p(T)$ is nilpotent for some irreducible polynomial $p$.

In the next two sections we treat separately the cases $p(t)=t$ (nilpotent $T$), and $p(t)$ different from $t$.

## 4. *The case of nilpotent T*

Theorem 2. *Suppose $T$ is nilpotent in the vector space $V$* [1]. *Let $N_1,\ldots$ be subspaces of $V$ and set $M_j=\sum_{i=0}^{j-1}T^iN_j$. Then*

(2) *each $M_j$ is a $t^j$-space with $N_j$ as $t^j$-base space and $V=\sum_j\oplus M_j=\sum_j\oplus\oplus\sum_{i=0}^{j-1}\oplus T^iN_j$*

*if and only if:*

(3) *for each $j$, $N_j$ is a value for*

$$\begin{aligned}&[N(T^j)-(N(T^j)\cap(N(T^{j-1})+TV))]\\ &=[N(T^j)-(N(T^{j-1})+(N(T^j)\cap TV))]\end{aligned}$$ [2].

Proof of (2) implies (3): [3] From (2), $N(T^j)$ is spanned by $\{T^iN_k \mid k\geqslant 1, i\geqslant\max(0,k-j)\}$ and $TV$ is spanned by $\{T^iN_k \mid k\geqslant 1, i\geqslant 1\}$. An easy calculation shows that $N_j\oplus(N(T^{j-1})+(N(T^j)\cap TV))=N(T^j)$ so (3) holds.

Proof of (3) implies (2): With $N_j$ as given in (3) we need only show:

(4) $T^{j-1}v\neq 0$ *if* $0\neq v\in N_j$;

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[1]) An elegant discussion by induction is given for the case of nilpotent $T$ (and hence, when $F$ is algebraically closed, for general $T$) by HALMOS [1, pages109–115] but our formulation gives (explicitly) information which is essential for the case of irreducible $p(t)$ different from $t$; in particular, the result of H. TROTTER [4] is included.

[2]) Equality here follows from the relation: $N(T^j)\supset N(T^{j-1})$ and the modular law.

[3]) If only the proof of Theorem 1 is wanted, then the proof of "(2) implies (3)" may be omitted.

(5) $\{T^iN_j \mid i \geqslant 0, j \geqslant 1\}$ *are independent*;

(6) $V = \sum_j \oplus \sum_{i=0}^{j-1} \oplus T^iN_j.$

Now (3) implies (4) since:

$$v \in N_j,\ T^{j-1}v = 0 \text{ imply } v \in N_j \cap N(T^{j-1}) = 0 \text{ by (3).}$$

Next, (3) implies (5); we need only show that for each $i \geqslant 0$:

(7) $T^{i+1}V, T^iN_1, T^iN_2, \ldots$ *are independent.*

Then it will follow that $\{T^iN_j \mid j \geqslant 1\}$ are independent for each $i \geqslant 0$, and $\{\sum_j T^iN_j \mid i \geqslant 0\}$ are independent since

$$(\textstyle\sum_j T^iN_j) \cap (\sum_{s>i}(\sum_j T^sN_j)) \leqslant (\sum_j T^iN_j) \cap T^{i+1}V = 0,$$

hence (5) holds.

To prove (7) suppose $T^iv_1 + \ldots + T^iv_s = T^{i+1}v$ with $v_j \in N_j$ for $j = 1, \ldots, s$. Then we must have $T^iv_s = 0$; for otherwise $i < s$ and left multiplication by $T^{(s-1-i)}$ yields $T^{s-1}v_s = T^sv$; then $v_s = (v_s - Tv) + Tv$ with $v_s \in N_s$ whereas $v_s - Tv \in N(T^{s-1})$ and $Tv \in N(T^s) \cap TV$ which implies $v_s = 0$, so $T^iv_s = 0$ after all. Repetition of this argument shows that all $T^iv_j = 0$ $(j = s, s-1, \ldots, 1)$ and hence $T^{i+1}v = 0$ so (7) is established.

Finally, (3) implies (6); we have for each $j \geqslant 1: N_j + TV + N(T^{j-1}) \geqslant N(T^j)$, hence $\sum_j N_j + TV \geqslant N(T^s) = V$ for $s = n^2$. Thus $\sum_j \sum_{i=0}^{j-1} T^iN_j + T^kV = V$ holds for $k = 1$, and then by induction (use left multiplication by $T$) for all $k$. When $k$ is sufficiently large, $T^k = 0$ and (6) is obtained.

Corollary. Theorem 1 holds if $T$ is nilpotent.

## 5. *The case $p(T)$ nilpotent with $p(t)$ irreducible and different from $t$*

In this section we suppose that $p(T)$ is nilpotent for some irreducible $p$ (of degree $m$, say) with $p(t)$ different from $t$ (then, by (xiii), $T$ is invertible).

Let $S = p(T)$. Then $S$ is nilpotent and we can apply section 4 to $S$. We set

$$N_j = [N(S^j) - (N(S^{j-1}) + (N(S^j) \cap SV))],$$

and

$$M_j = \sum \oplus (S^iN_j \mid 0 \leqslant i < j).$$

We assert:

Lemma 1. *Suppose that for each $j$, $N_j$ can be so chosen* [1]) *that for some $\bar{N}_j$:*

$$N_j = \bar{N}_j \oplus T\bar{N}_j \oplus \ldots \oplus T^{m-1}\bar{N}_j.$$

*Then $M_j$ is a $p^j$-$T$-space with $\bar{N}_j$* as *$p^j$-base space.*

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[1]) There are, in general, different possible values for the relative complement $[N(S^j) - (N(S^{j-1}) + (N(S^j) \cap SV))]$.

Proof. We need to show that for fixed $j$:

(8) $T^i\bar{N}_j$, $i=0, 1, \ldots, mj-1$ *are equidimensional*;

(9) $M_j = \sum_{i=0}^{mj-1} \oplus T\bar{N}_j$;

(10) $p^j(T)(\sum_{i=0}^{mj-1} T^i\bar{N}_j)=0.$

But (8) follows from the fact that $T$ is invertible.

And (9) follows because

$$M_j = \sum_{i=0}^{j=1} \oplus (p^i(T)N_j) \subset \sum_{i=0}^{\infty} T^i\bar{N}_j = \sum_{i=0}^{mj-1} T^i\bar{N}_j,$$

$D(M_j)=jD(N_j)=mjD(\bar{N}_j)$, and for all $i$, $D(T^i\bar{N}_j) \leqslant D(\bar{N}_j)$; this forces (9) to hold.

Finally, (10) follows from the fact that $\bar{N}_j \subset N_j$ and $p^j(T)\,N_j=0$, hence $p^j(T)\,T^i\,\bar{N}_j = T^i\,p^j(T)\,\bar{N}_j=0$.

Remark. If $N_j$ and $\bar{N}_j$ can be chosen for each $j$ to satisfy the hypotheses of Lemma 1 then Theorem 1 (a) follows immediately. The essential complication in the proof of the structure theorem (as we present it) is precisely to show that such $N_j$, $\bar{N}_j$ exist. This complication can be disposed of easily but an "existence" type of argument seems to be required.

Theorem 3. *Let $j$ be fixed and suppose $T$, $p$, $S$ are as described above. Suppose $Y$ is a subspace such that*: $N(S^j) \supset Y \supset (N(S^{j-1}) + (N(S^j) \cap SV))$, $TY \subset Y$ *and* $Y \neq N(S^j)$.

*Then there exists some subspace $X$ satisfying*:

(*) $X \neq 0$,

(**) $T^mX \subset X \oplus TX \oplus \ldots \oplus T^{m-1}\,X \oplus Y \subset N(S^j)$.

Proof. Let $v$ be any non-zero vector satisfying $v \in N(S^j)$ but $v \notin Y$, let $h(t)$ be a polynomial of leading coefficient 1 and least degree such that $h(T)v \in Y$ and set $X=[v]$.

If (degree of $h$)<(degree of $p$) then $h$, $p$ are relatively prime so $h(T)$ is invertible (by (xiii)) and $v=(h(T))^{-1}\,h(T)\,v \subset h(T)\,Y \subset Y$, a contradiction. So (degree of $h$)$\geqslant$(degree of $p$). This implies that $X, TX, \ldots, T^{m-1}X$, $Y$ are independent and that (*), (**) hold.

Corollary 1. There exist $N_j$, $\bar{N}_j$ satisfying the hypotheses of Lemma 1.

Proof. Let $Y_0 = N(S^{j-1}) + (N(S^j) \cap SV)$.

If $Y_0 = N(S^j)$ we may take $N_j=\bar{N}_j=0$.

If $Y_0 \neq N(S^j)$ then Theorem 3 can be applied with $Y=Y_0$. Let $X_0$ be a *maximal* $X$ with properties (*), (**) (with $Y=Y_0$) and let

$$Y_1 = X_0 \oplus TX_0 \oplus \ldots \oplus T^{m-1}X_0 \oplus Y_0.$$

Now $Y_1 \subset N(S^j)$ so if $Y_1 \neq N(S^j)$, Theorem 3 can be applied with $Y=Y_1$ to obtain $X=X_1 \neq 0$, satisfying (*), (**) (with $Y=Y_1$). Then $X_0 \oplus X_1$ satisfies (*), (**) (with $Y=Y_0$) and $X_0 \oplus X_1 \neq X_0$ contradicting the maximality of $X_0$.

Thus $Y_1 = N(S^j)$ and it suffices to choose $\bar{N}_j = X_0$ and $N_j = X_0 \oplus TX_0 \oplus \dots$ $\dots \oplus T^{m-1} X_0$ to establish Corollary 1.

Corollary 2. Theorem 1 holds for the case $p(T)$ nilpotent ($p$ irreducible and different from $t$) and hence Theorem 1 is completely proved.

## 6. *Extensions of the structure theorem*

In this section $\mathfrak{R}$ will denote an associative ring with unit and $F$, the centre of $\mathfrak{R}$, will be assumed to be a division ring. $V=\{0, v, \dots\}$ will denote a right module over $\mathfrak{R}$, that is, an abelian group for which $v\alpha$ is defined and in $V$ for all $v$ in $V$, $\alpha$ in $\mathfrak{R}$, and satisfies the identities: $(v_1+v_2)\alpha = v_1\alpha + v_2\alpha$, $v(\alpha_1+\alpha_2) = v\alpha_1 + v\alpha_2$, $v(\alpha_1\alpha_2) = (v\alpha_1)\alpha_2$, $v1 = v$. If $\alpha$ is in the centre of $\mathfrak{R}$ we interpret $\alpha v$ to mean $v\alpha$.

$T$ will denote a linear transformation in $V$. This means: $Tv$ is defined and in $v$ for all $v$ in $V$, and $T(v_1+v_2) = Tv_1 + Tv_2$, $T(v\alpha) = (Tv)\alpha$.

As in sections 1–5 we consider only polynomials $q(t)$ which have coefficients in $F$ so $q(T)$ is a linear transformation along with $T$.

Generalizing (xv) we call $M$ a *q-module* with $N$ as *q-base module* if $q$ (of degree $m$, say) is a power of an irreducible polynomial and $M$, $N$ are submodules of $V$ such that: $q(T)=0$, $M = N \oplus TN \oplus \dots \oplus T^{m-1}N$, and $v \in N$, $T^{m-1}v = 0$ together imply $v=0$.

Now we ask: under what additional conditions do the theorems of sections 1–5 (with submodulus in place of vector subspace) continue to hold?

It is easy to see that Theorem 1(i) for nilpotent $T$, (that is, Theorem 2) holds if and only if

(11) relative complements $[N(T^j) - (N(T^j) \cap (N(T^{j-1}) + TV))]$ exist for all $j$.

It can also be seen that Theorem 1 (i) holds when $S (\equiv p(T))$ is nilpotent with $p$ irreducible and different from $t$ providing:

$\overline{\text{(i)}}$ : (11) holds with $S$ in place of $T$;

$\overline{\text{(ii)}}$ : Whenever $A_1, \dots, A_r$, $B_1, \dots, B_r$ are submodules such that $A_1 \oplus \dots \oplus A_r \subset B_1 + \dots + B_r$ and each $A_i (i > 1)$ and each $B_i (i \geqslant 1)$ is a (1,1) map of $A_1$ by some polynomial in $T$, then $B_1, \dots, B_r$ are independent, and $\subset$ can be replaced by $=$;

$\overline{\text{(iii)}}$: Theorem 3 holds and has some maximal solution $X$.

Finally, if Theorem 1 (a) holds when $p(T)$ is nilpotent (for every irreducible $p$) then Theorem 1 (a) holds whenever $h(T) = 0$ for some polynomial $h$ (then $T$ is called algebraic).

It can be shown (see [2] for terminology and proofs): if $T$ is algebraic then condition (11) will be satisfied with $p(T)$ in place of $T$ (for every irreducible polynomial $p$) if $\mathfrak{R}$ is regular (in von Neumann's sense) and $N(h(T))$ is finitely generated for every polynomial $h$.

The remaining condition $\overline{\text{(ii)}}$ above (needed only for finitely generated $A_1$) will then be verified if $\bar{R}_{\mathfrak{R}}$, the lattice of principal right ideals of $\mathfrak{R}$ ($\bar{R}_{\mathfrak{R}}$ is necessarily complemented and modular), is countably complete and countably continuous; and the remaining condition $\overline{\text{(iii)}}$ will be satisfied if $\bar{R}_{\mathfrak{R}}$ is a von Neumann ring (necessarily irreducible since the centre $F$ is assumed to be a division ring).

Discussions of Theorem 1 (b), (c), and extensions to reducible $\mathfrak{R}$ and to almost-algebraic $T$, can be derived from the purely ring-theoretic discussion given in [2]; we omit the details.

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## REFERENCES

1. HALMOS, P. R., Finite-Dimensional Vector Spaces, (second edition), Van Nostrand (New York, 1958).
2. HALPERIN, ISRAEL, Elementary divisors in von Neumann rings, Acta Scientiarum Mathematicarum Szeged, to appear.
3. JACOBSON, N., Linear Algebra, (van Nostrand, New York, 1953).
4. TROTTER, H. F., A canonical basis for nilpotent transformations, American Math. Monthly, **68**, 779–780 (1961).
5. WAERDEN, L. VAN DER, Modern Algebra, vol. II. (Ungar, New York, 1950).